Discretization: Calculating the Z-transform of LTI systems modelled with Laplace Transform

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I. Introduction

This report shows how to obtain the Z transform of a given Laplace transform, using Tustin method. The same procedure is repeated a number of tomes, for different specific transfer functions.

II. Discretization of a PI regulator.

The first case of study is for a typical PI regultar. If a discrete PI controller needs to be calculated from the , s-domain Laplace transform of a controller, firtsly this expression must be calculated:

$$GR(s) = Kp \cdot \left(1 + \frac{1}{T_I \cdot s}\right) \quad (1)$$

For simplicity, the expression will be defined considering $K_I = \frac{1}{T_I}$, and therefore:

$$GR(s) = Kp \cdot \frac{s+K_I}{s} \quad (2)$$

The discrete expression of the regulator, in the z domain will be calculated. In order to do that, the relationship between the transform variables s and z, considering a given sampling time, T_m, needs to be defined. In this case, the Tustin transform will be assumed, and therefore:

$$s = \frac{2}{T_m} \cdot \frac{z-1}{z+1} \quad (3)$$

Now, this expression is substituted in equation (2):

$$GR = Kp \cdot \frac{s+K_I}{s} = Kp \cdot \frac{\frac{2}{T_m} \cdot \frac{z-1}{z+1} + K_I}{\frac{2}{T_m} \cdot \frac{z-1}{z+1}} =$$

$$= Kp \cdot \left(1 + K_I \cdot \frac{T_m}{2} \cdot \frac{z+1}{z-1}\right) \quad (4)$$

This expression is now decomposed in fractions:

$$\frac{r_n}{e_n}(z-1) = Kp \cdot \left((z-1) + K_I \frac{T_m}{2}(z+1)\right) \quad (5)$$

$$\frac{r_n}{e_n}(1-z^{-1}) = Kp \cdot \left((1-z^{-1}) + K_I \frac{T_m}{2}(1+z^{-1})\right) \quad (6)$$

$$\frac{r_n}{e_n} - \frac{r_n}{e_n} z^{-1} =$$

$$= Kp \cdot \left[\left(K_I \cdot \frac{T_m}{2} + 1\right) + \left(K_I \cdot \frac{T_m}{2} - 1\right) \cdot z^{-1}\right] \quad (7)$$

$$\frac{r_n}{e_n} = Kp\left[\left(K_I \frac{T_m}{2} + 1\right) + \left(K_I \frac{T_m}{2} - 1\right) z^{-1}\right] + \frac{r_n}{e_n} z^{-1} \quad (8)$$

$$r_n = Kp\left[\left(K_I \frac{T_m}{2} + 1\right) + \left(K_I \frac{T_m}{2} - 1\right) z^{-1}\right] e_n + r_n \cdot z^{-1} \quad (9)$$

$$r_n = Kp\left[\left(K_I \frac{T_m}{2} + 1\right) e_n + \left(K_I \frac{T_m}{2} - 1\right) e_{n-1}\right] + r_{n-1} \quad (10)$$

This gives the final expression of the z transform:

III. Discretization of a P regulator.

El regulador proporcional es más sencillo, a saber:

$$GR = Kp = \frac{r_n}{e_n} \quad (11)$$

Por tanto

$$r_n = Kp \cdot e_n \quad (12)$$

IV. Transformada en Z: Ecuación de un regulador PID.

Si queremos realizar un control PID, debemos utilizar un regulador como el siguiente:

$$GR(s) = K \cdot \left(1 + \frac{1}{T_I \cdot s} + T_d \cdot s\right) \quad (13)$$

(ojo, derivador ideal, no se puede implementar en la práctica).

$$GR(z) = K \cdot \left(1 + \frac{T_m}{2 \cdot T_I} \cdot \frac{1+z^{-1}}{1-z^{-1}} + \frac{T_d}{T_m} \cdot (1-z^{-1})\right) \ldots (14)$$

V. Second Order Low Pass Filter

$$G(s) = K \cdot \left(\frac{\omega_C^2}{s^2 + 2 \cdot \xi \cdot \omega_C \cdot s + \omega_C^2}\right) \quad (15)$$

Se pretende calcular cómo tiene que ser la ecuación en diferencias para implementar este regulador en digital.

Mediante la transformación de Tustin, $s = \frac{2}{T_m} \cdot \frac{z-1}{z+1}$, puede calcularse esta expresión.

$$GR = K \cdot \left(\frac{\omega_C^2}{\left(\frac{2}{T_m} \cdot \frac{z-1}{z+1}\right)^2 + 2 \cdot \xi \cdot \omega_C \cdot \frac{2}{T_m} \cdot \frac{z-1}{z+1} + \omega_C^2}\right) = \cdots$$

$$= Ko \cdot \frac{z^2 + 2z + 1}{z^2 \cdot x_2 + 2z \cdot x_1 + x_0} \quad (16)$$

donde

$$Ko = K \cdot \omega_C^2 \quad (17)$$

$$x_0 = \frac{4}{T_m^2} - \frac{4 \cdot \xi \cdot \omega_C}{T_m} + \omega_C^2 \quad (18)$$

$$x_1 = 2 \cdot \omega_C^2 - \frac{8}{T_m} \quad (19)$$

$$x_2 = \frac{4}{T_m^2} + \frac{4 \cdot \xi \cdot \omega_C}{T_m} + \omega_C^2 \quad (20)$$

La ec en diferencias es:

$$\frac{o_n}{i_n} \cdot (x_2 \cdot z^2 + x_1 \cdot z + x_0) = Ko \cdot (z^2 + 2 \cdot z + 1) \quad (21)$$

$$o_k \cdot (x_2 + x_1 \cdot z^{-1} + x_0 \cdot z^{-2}) =$$

$$= Ko \cdot i_k \cdot (1 + 2 \cdot z^{-1} + z^{-2}) \qquad (22)$$

Y, finalmente,

$$o_k = \frac{1}{x_2} \cdot (Ko \cdot i_k + 2 \cdot Ko \cdot i_{k-1} + Ko \cdot i_{k-2} - x_1 \cdot o_{k-1} - x_0 \cdot o_{k-2}) \qquad (23)$$

donde

$$Ko = K \cdot {\omega_C}^2 \qquad (24)$$

$$x_0 = \frac{4}{{T_m}^2} - \frac{4 \cdot \xi \cdot \omega_C}{T_m} + {\omega_C}^2 \qquad (25)$$

$$x_1 = 2 \cdot {\omega_C}^2 - \frac{8}{T_m} \qquad (26)$$

$$x_2 = \frac{4}{{T_m}^2} + \frac{4 \cdot \xi \cdot \omega_C}{T_m} + {\omega_C}^2 \qquad (27)$$